The Bandwidth Dependence of Time Domain Frequency Stability Measurements

W.J. Riley
Hamilton Technical Services
Beaufort, SC 29907 USA
May 29, 2020

The Allan deviation stability of a frequency source that is dominated by white (and to a lesser extent, flicker) phase noise depends on the bandwidth of the overall system. There is always an upper bandwidth limit in any real situation. In most cases that bandwidth is large enough that the ADEV follows a -1 slope over its full tau range and varies as the square root of the white PM noise bandwidth. The latter may depend on the source (e.g., a crystal bandpass filter in the RF output), a subsequent phase locked loop (PLL), or the measuring system (e.g., the low pass filter following a mixer). If the source PM noise and resulting ADEV is being modeled, or is being converted from frequency domain measurements, then one must use the value of the system bandwidth for the purposes of the model or conversion [3].

More specifically, the expression that determines the effect of the system bandwidth for time domain frequency stability measurements is $2\pi f_h \tau$, where f_h is the (usually assumed abrupt) system high frequency cutoff frequency and τ is the measurement interval or analysis averaging time. If $2\pi f_h \tau >> 1$, the usual case, the system bandwidth is large enough that the ADEV plot follows a constant -1 slope. The relevant expressions for white and flicker PM noise are shown in the table below (see References [1] and [2]. Table 1 and Figure 5 of Reference [1] are reproduced below as Figures 1 and 2.

Table 1 Listing of the five common types of noise found in practical sources. The asymptotic forms of $\sigma_y^2(\tau)$ for various power-law noise types and two filter types are also listed. Note: $\omega_h/2\pi = f_h$ is the measurement system bandwidth—often called the high-frequency cutoff. ln $\equiv$ log

			$\sigma_y^2(\tau)$			
			$\omega_h \tau >> 1$	$\omega_h \tau >> 1$	$\omega_h \tau << 1$	$\omega_h \tau << 1$
Name of Noise	α	$S_y(f)$	Infinite Sharp Filter	Single Pole Filter	Infinite Sharp Filter	Single Pole Filter
White phase	2	$h_2 f^2$	$\frac{3 f_h h_2}{(2\pi)^2 \tau^2}$	$\frac{3 f_h h_2}{(2\pi)^2 \tau^2}$	$2/5\pi^2 f_h^5 \tau^2 h_2$	$\frac{f_h^2 h_2}{2\tau}$
Flicker phase	1	$h_1 f$	$\frac{(1.038+3\ln(\omega_h \tau))h_1}{(2\pi)^2 \tau^2}$	$\frac{(3\ln(\omega_h \tau))h_1}{(2\pi)^2 \tau^2}$	$1/2\pi^2 f_h{}^4 \tau^2 h_1$	$2 f_h^2 (\ln(2)) h_1$
White frequency	0	h_o	$\frac{h_o}{2\tau}$	$\frac{h_o}{2\tau}$	$2/3\pi^2 f_h{}^3 \tau^2 h_0$	$2/3\pi^2 f_h{}^2 \tau h_0$
Flicker frequency	-1	$h_{-1} f^{-1}$	$2(\ln(2))h_{-1}$	$2(\ln(2))h_{-1}$	$\pi^2 f_h{}^2 \tau^2 h_{-1}$	$8\pi^2 f_h{}^2 \tau^2 h_{-1}$
Random-walk frequency	-2	$h_{-2} f^{-2}$	$\frac{2\pi^2 \tau h_{-2}}{3}$	$\frac{2\pi^2 \tau h_{-2}}{3}$	$2\pi^2 f_h \tau^2 h_{-2}$	$2\pi^2 f_h \tau^2 h_{-2}$

Figure 1. Allan Variance Expressions for Common Noise Types

In Figure 5 of Reference [1] (Figure 2 herein) you will notice that both the level and shape of the ADEV plot for white PM noise can depend on the system bandwidth, f_h. But you will also notice that, in order to make a difference to the shape, the measurement bandwidth has to be very low, orders of magnitude below 1 Hz, even for the case of a sharp filter.

The ADEV level depends on the PM noise density and the system noise bandwidth. Consider the case of a 10 MHz frequency source having white phase noise at -80 dBc/Hz. If the system BW varies from 1, 10 and 100 Hz, the 1-second ADEV will vary from 3.9e-12, 1.2e-11 and 3.9e-11 respectively.

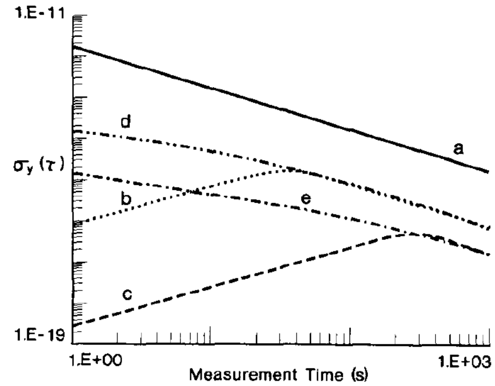


Fig. 5. $\sigma_y(\tau)$ for white phase modulation ($\alpha = 2$) as a function of measurement time, τ, and measurement bandwidth, f_h. Curves a, b, and c have an infinitely sharp filter with width, $f_h = 16$ Hz, $f_h = 0.016$ Hz, $f_h = 0.0016$ Hz, respectively. Curves d and e have a single pole filter width, $f_h = 0.016$ Hz and $f_h = 0.0016$, respectively. $h_2 = 2 \times 10^{-24}$

Figure 2. ADEV versus Measurement BW

Rutman and Walls (1991) Figure 5 showing the ADEV for filtered white phase noise with various noise bandwidths can be reproduced in Stable32 as shown in Figure 3 by simulating white phase noise data at the three nominal W PM levels and then using the low pass Filter function. The f_h=16 Hz case is unfiltered since that is well above the 0.5 Hz Fourier frequency limit of the 1 Hz data. We see that the ADEV is underestimated for $\tau < (2 \cdot f_h)^{-1}$.

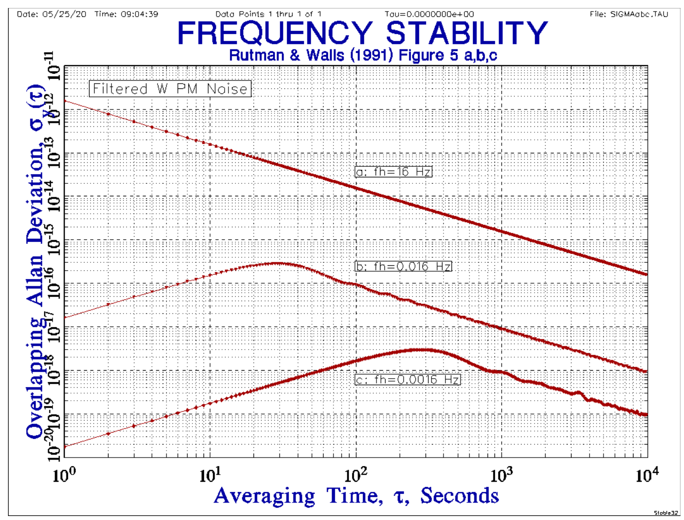


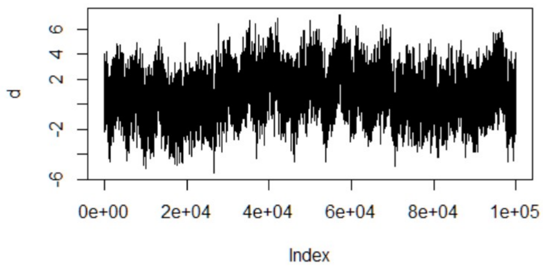
Figure 3. ADEV versus Measurement BW Using Stable32

The canonical heterodyne clock measurement system comprises an offset local oscillator, RF mixer, low pass filter, zero-crossing detector and counter, often configured as a dual mixer time difference configuration, perhaps using cross-correlation for a lower noise floor [9].

Aliasing occurs when the system bandwidth is greater than one-half of the sampling rate, the usual condition for those clock measuring systems. But aliasing is not an issue when dealing with a flat white PM noise spectrum without discrete components, and the energy and variance of the noise is not affected by aliasing [4]. In the case of flicker PM noise, aliasing can produce a spurious white PM noise component (see [5] Part II, Section 2.3). One can observe this as shown in Figure 5 by generating a large sample of F PM noise (confirming its α=+1 nominal ACF noise fit, β=-1 nominal PSD fit, and -1 nominal MDEV slope), and then "average" (downsample with aliasing) it by a large factor (e.g., AF=100) and observing that the noise becomes dominantly W PM ($\alpha \approx +2$, $\beta \approx 0$, MDEV slope≈-1.5). This can be a source of confusion. If the noise lowpass filtration at the lower Nyquist rate is done before downsampling (i.e., proper decimation, often done in stages), the F PM noise characteristic and original ADEV is retained at the lower tau [13].

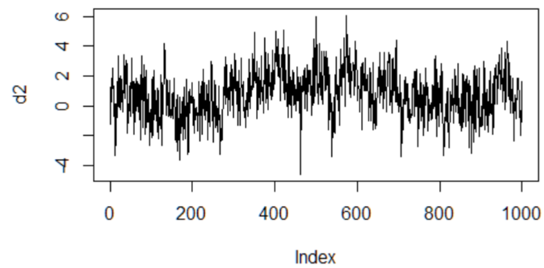
The original 100,000 points of flicker PM noise below with τ=0.01s and $\sigma_y(t)$=1.0, have an α=+1.00, β=-0.98 and -1.0 nominal MDEV slope.

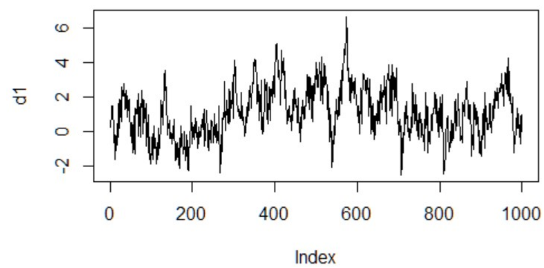
The 1000 points with τ=1s downsampled without lowpass filtering at the upper right have an α=+1.72, β=-0.85 and -1.5 nominal MDEV slope and are changed to dominantly white PM by aliasing.



The 1000 points with τ=1s properly decimated at the lower right have ACF α=+0.66 , β=-1.27 and -1.0 nominal MDEV slope are still dominantly flicker PM noise.

Downsampled x100





Decimated x100

Figure 5. Comparison of Downsampling and Decimation for Flicker PM Noise

One almost never needs to worry about white PM noise departure from its nominal -1 ADEV slope since the bandwidth is wide and the $2\pi f_h\tau >> 1$ case applies. But the ADEV will scale with the square root of the system noise bandwidth, as will the related rms phase jitter and residual FM. Thus, for a source with significant white or flicker PM noise, one should always note the system bandwidth when performing a time domain frequency stability measurement.

The frequency response of AVAR (see Figure 6), determined by the Fourier transform of its sampling function, looks like a ½-octave-wide band pass filter. The peak in the response is at f = $0.5/\tau_0$, where f is a Fourier component of fractional frequency deviation y(t) and τ_0 is the basic sampling time of the frequency data. Because there is considerable energy in the sidelobes (the 2nd one is only about -10 dB) one should not restrict the measurement bandwidth to less than about x2 the peak response, or about $(\tau_0)^{-1}$. That precludes using an effective anti-aliasing filter with the usual heterodyne/zero-crossing detector type clock measuring system.

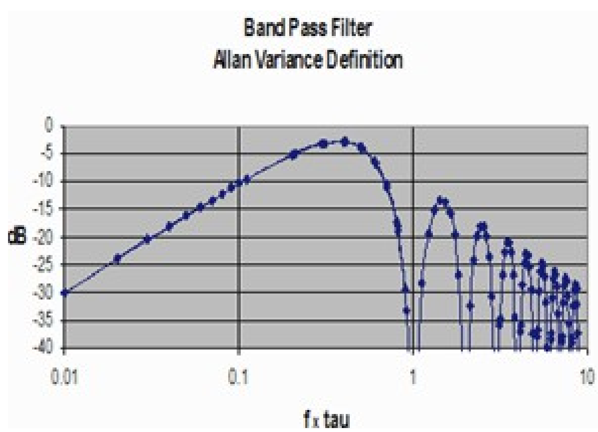


Figure 6.
Frequency Response of the AVAR (From [4]).

Reference [5] contains further useful insight into ADEV sampling and system bandwidth considerations. One technique described therein was to make measurements of a white phase noise dominated source with both the standard and modified Allan variances where AVAR has the upper cutoff frequency (bandwidth) dependence discussed above and (without aliasing) MVAR does not. The MVAR h_2 value was used along with the AVAR $f_h \cdot h_2$ product to determine the system bandwidth f_h. It was found to be equal to one-half of the sampling interval. This finding does not, of course, provide information about the bandwidth of the measuring system prior to the sampling process. That presumably could be determined by making a separate frequency domain phase noise measurement, calculating h_2 and then f_h. A more elaborate method for determining the time domain measuring system bandwidth is described in Section III-E of References [10] and [12], where an adjustable amount of bandpass-filtered white PM noise is added to an RF carrier, along with a clean RF output as a coherent reference. A similar phase noise standard is described in Reference [11].

Reference [5] finds that spectral aliasing is noticeable only for white and flicker PM noise, and that one must distinguish between the effects of spectral aliasing and the normal variance bandwidth dependence. Spectral aliasing does not affect ADEV. They conclude that correct estimation of the white phase noise levels of a sampled signal may be achieved with variances even if the sampling frequency is far lower than the high cut-off frequency and that the effects of spectral aliasing for low-frequency noises may be neglected.

Another artifact of data sampling associated with ADEV, one that causes quasi-oscillatory behavior at long tau, is discussed in Reference [6]. This effect can result from interference between the ADEV sampling response and long-period divergent noise, because the number of X^2 degrees of freedom is small, by data turn-on "ringing", or because of leakage from wideband PM noise. Those effects are independent from issues of system bandwidth and aliasing, and the solution is to use the more capable Total and Thêo1 estimators.

The advent of direct digital clock measurement instruments has created a way to sample and process phase data for both time and frequency domain analysis without aliasing [7], [8]. Those instruments replace the classic heterodyne mixer, low pass filter and zero-crossing detector with an anti-alias filter, RF sampler followed by I/Q digital downconversion, stages of decimation, and a digital phase detection ($\tan^{-1}$) similar to a modern software-defined radio (SDR). There is no aliasing, and the noise bandwidth is well-defined. They are versatile and have high performance, but complex and expensive, and there is still a place for the traditional analog techniques.

The conclusion for a traditional time domain analog clock measurement system is that the knowledge about system bandwidth is needed to find the white and flicker phase noise level from ADEV values, and that that measurement parameter should be noted, but that aliasing caused by the non-Nyquist sampling does not affect an ADEV analysis (see [5] Part II, Table 3).

References:

1. J. Rutman and F.L. Walls, "Characterization of Frequency Stability in Precision Frequency Sources," *Proceedings of the IEEE*, Vol. 79, No. 6, June 1991. The information in Table 1 appears widely in the frequency stability analysis literature. Figure 5 also appears in a September 1991 monograph describing the NIST SIGINT computer program for converting frequency domain stability measurements into time domain values which included F.L. Walls as one of its authors, Reference [28] of this paper.
2. W.J. Riley, *Handbook of Frequency Stability Analysis*, NIST Special Publication 1065, July 2008. The last two terms of Equation (67) in Section 7 on page 73 contain the same expressions for flicker and white phase noise.
3. The Domain function of the Stable32 program for frequency stability analysis includes the system bandwidth factor for the cases of white and flicker PM noise.
4. W.J. Riley, "The Averaging of Phase and Frequency Data," Stable32 web site, November 2011.

5. F Vernotte, G. Zalamansky and E Lantz, "Time Stability Characterization and Spectral Aliasing," Parts I and II", *Metrologia*, Vol. 35, No. 5, pp. 723-738, October 1998. These papers deserve more attention.
6. D. Howe, "Interpreting Oscillatory Frequency Stability Plots," *Proceedings of the 2002 IEEE International Frequency Control Symposium*, pp. 725-732, May 2002.
7. S.R. Stein, "The Allan Variance – Challenges and Opportunities," Symmetricom, Inc.
8. J. Grove et al, "Direct Digital Measurement of Precision Oscillators," *Proceedings of the 2004 IEEE International Frequency Control Symposium*, pp. 287-291, August 2004, and , S.R. Stein, Microsemi, July 2017.
9. W.J. Riley, "Time and Frequency Measurements," Stable32 web site, April 2019.
10. F.L. Walls, "Secondary Standard for PM and AM Noise at 5, 10, and 100 MHz," *IEEE Transactions on Instrumentation and Measurement*, Vol. 42, No. 2, April 1993.
11. K. Watabe, S. Yanagimachi, T. Ikegami, H. Iida and Y. Shimada, "−100 dBc/Hz Flat Phase Noise Signal at 10 MHz for Phase Noise Standards," *Proceedings of the 2012 Conference on Precision Electromagnetic Measurements*, pp. 620-621, July 2012.
12. U.S. Patent No. 5,172,064, "Calibration System for Determining the Accuracy of Phase Modulation and Amplitude Modulation Noise Measurement Apparatus," December 1992. This patent contains detailed information about the design, calibration, and use of the PM and AM noise standard of Reference [10], particularly the information following Eq. (9) related to time domain frequency stability measurements.
13. This can be explored in R with the downsample() function of R. Everett, "A simple downsample function for vectors in R", March 2011, and the decimate() function in the CRAN 'signal' package.